Problem Sheet 2:Additional Questions v2

Problem Sheet 2:Stirling's Theorem

Stirling's Theorem relates to any asymptotic result for $N!$ as $N \to \infty$. In Problem Sheet 1 you were asked to show

$$e\left(\frac{N}{e}\right)^N \le N! \le eN\left(\frac{N}{e}\right)^N, \tag{35}$$

by examining

$$\log N! = \sum_{1\le n\le N} \log n.$$

In Chapter 2 we showed that

$$\sum_{1\le n\le x} \log x = x\log x - x + O(\log x)\,, \tag{36}$$

for any *real* $x > 1$. To improve (35) we need to improve (36), but only for *integer* x.

17. The function $\{t\}$ is periodic, period 1. The result of Question 1

$$\int_\alpha^{\alpha+1} \{t\}\,dt = \frac{1}{2}, \tag{37}$$

can be interpreted as saying that $\{t\}$ has average value 1/2.

Define

$$P(x) = \int_1^x \left(\{t\} - \frac{1}{2}\right) dt.$$

Prove that $P(x)$ is periodic, period 1 and $P(n) = 0$ for all $n \in \mathbb{Z}$.

18. *Euler Summation, Proposition 2.8, is for sums over $n \le x$ with x* ***real****. Improved results can be given when x is an* ***integer****.*

i) Prove

$$\sum_{1\le n\le N} \log n = N\log N - N + 1 + \int_1^N \frac{\{t\}}{t} dt, \tag{38}$$

for *integers* $N \ge 1$.

ii) Prove, by integrating by parts,

$$\sum_{1 \leq n \leq N} \log n = N \log N - N + \frac{1}{2} \log N + 1 + \int_1^N \frac{P(t)}{t^2} dt, \tag{39}$$

for *integers* $N \geq 1$, where $P(t)$ is the function seen in Question 17.

iii) Prove that there exists a constant C such that

$$\sum_{1 \leq n \leq N} \log n = N \log N - N + \frac{1}{2} \log N + C + O\left(\frac{1}{N}\right), \tag{40}$$

for *integers* $N \geq 1$.

Hint Use an idea seen in the proof of Theorem 2.9, replacing any integral over $[1, x]$ of an integrable function by an integral over $[1, \infty)$ and then estimating the tail end integral over (x, ∞).

This could be compared with (36) ,

$$\sum_{1 \leq n \leq x} \log n = x \log x - x + O(\log x) ,$$

for real $x \geq 1$.

iv) Deduce *Stirling's formula* in the form

$$N! = A \left(\frac{N}{e}\right)^N \sqrt{N} \left(1 + O\left(\frac{1}{N}\right)\right),$$

for some constant A.

It can be shown that $A = \sqrt{2\pi}$ (but not here)..

This shows that the true result for $N!$ lies 'midway' between the bounds in (35).

19. *For those who need more like Questions 11 and 12.* On a previous Problem Sheet you were asked to show that

$$\sum_{n=3}^{\infty} \frac{1}{n \log n \left(\log \log n\right)^{\beta}}$$

converges iff $\beta > 1$? For various reasons we consider the n-th prime to satisfy $p_n \approx n \log n$. Then $\log p_n \approx \log n$ and

$$p_n \left(\log \log p_n\right)^\beta \approx n \log n \left(\log \log n\right)^\beta .$$

Prove that

$$\sum_{p \geq 3} \frac{1}{p \left(\log \log p\right)^\beta}$$

converges iff $\beta > 1$.

Hint Remove the $1/\left(\log \log p\right)^\beta$ factor by partial summation and then apply Merten's Theorems.

20. i) Using the same method as in Question 18 prove that

$$\sum_{1 \leq n \leq N} \frac{1}{n} = \log N + \gamma + \frac{1}{2N} + O\left(\frac{1}{N^2}\right), \tag{41}$$

for *integer* $N \geq 1$, and where γ is Euler's constant.

ii) Why will (41) not hold if the integer N is replaced by *real* x?

In previous questions we have looked at $\sum \left(\log n\right)^\ell$ and $\sum \left(\log n\right)^\ell / n$. The latter sum has a distinctly simpler result, two main terms and best possible error, whilst the first has $\ell + 1$ main terms. Here we look at $\sum \log^\ell \left(x/n\right)$ and $\sum \left(\log^\ell \left(x/n\right)\right)/n$. The results are reversed in that the first sum has a simpler form than the second.

21. Prove, using (29), that for all integers $\ell \geq 1$ we have

$$\sum_{n \leq x} \frac{\log^\ell \left(x/n\right)}{n} = \frac{1}{\ell+1} \log^{\ell+1} x + O\left(\log^\ell x\right).$$

22. Use Question 5 to improve Question 21 and show that for every integer $\ell \geq 1$, there exists a polynomial $Q_\ell\left(y\right)$ of degree $\ell+1$ leading coefficient $1/\left(\ell+1\right)$ such that

$$\sum_{n \leq x} \frac{\log^\ell \left(x/n\right)}{n} = Q_\ell\left(\log x\right) + O\left(\frac{\log^\ell x}{x}\right).$$

Hint use the Binomial expansion on $\log^\ell \left(x/n\right)$.

23. Apply (29) to prove that

$$\sum_{n\le x} \log^{\ell}\left(\frac{x}{n}\right) = \ell! x + O_{\ell}\big(\log^{\ell} x\big)\,, \tag{42}$$

for all integers $\ell \ge 0$.

24. Prove the **discrete** version of Partial Summation: For integers $N \ge M \ge 1$ we have

$$\begin{aligned}\sum_{r=M}^{N} a_r f(r) &= \sum_{r=M}^{N-1} A(r)\,(f(r) - f(r+1)) \\ &\quad + A(N)\,f(N) - A(M-1)\,f(M)\,,\end{aligned}$$

where $A(n) = \sum_{r=1}^{n} a_r$ (Conventionally $A(0) = 0$).

Note This result is useful when f is **not** differentiable. If f has a continuous derivative you can write

$$f(r) - f(r+1) = -\int_r^{r+1} f'(t)\,dt,$$

and you recover the Partial Summation seen in the notes.

Hint Note that $a_r = A(r) - A(r-1)$.

25. Prove **Dirichlet's test for convergence.** In the notation of Question 24, suppose that

i) there exists $C > 0$ such that $|A(r)| \le C$ for all $r \ge 1$;

ii) $f(r) \to 0$ as $r \to \infty$;

iii) $\sum_{r=1}^{\infty} |f(r) - f(r+1)|$ is convergent, with sum F, say.

Then $\sum_{r=1}^{\infty} a_r f(r)$ converges, to S say, with $|S| \le CF$.

Hint Use partial summation to rewrite the given sum as a sum on which you can apply a comparison test for series from First Year Analysis.

26. Question on Merten's Theorems extended Recall from Chapter 1 that given $N \in \mathbb{N}$ the set $\mathcal{N}$ is defined by $\mathcal{N} = \{n : p|n \Rightarrow p \leq N\}$ and we used the fact that

$$\sum_{n\in\mathcal{N}} \frac{1}{n} > \sum_{n\leq N} \frac{1}{n} = \log N + O(1) \,.$$

Prove that

$$\sum_{n\in\mathcal{N}} \frac{1}{n} = \kappa \log N + O(1)$$

for some $\kappa > 0$.

What is the numerical value of κ?

Hint Write this sum as an Euler Product.

27. (Hard) Show that if $h(t) \to c$ as $t \to \infty$ then

$$\lim_{x\to\infty} \frac{1}{\log x} \int_1^x \frac{h(t)}{t} dt \to c.$$

Deduce from Question 14 that IF $\lim_{t\to\infty} \psi(t)/t$ exists then the limit has value 1.

Note, this is *not* a proof of the Prime Number Theorem, but shows what the correct statement of the Prime Number Theorem should be.

Hint You need to verify the $\varepsilon - X$ definition of limit at infinity, i.e. for all $\varepsilon > 0$ there exists X such that if $x > X$ then

$$\left| \frac{1}{\log x} \int_1^x \frac{h(t)}{t} dt - c \right| < \varepsilon.$$

To this end write

$$c = \frac{1}{\log x} \int_1^x \frac{c}{t} dt,$$

substitute in and try to make use of the assumption that $h(t) \to c$ as $t \to \infty$.